Math 55 Midterm Exam #1

Summer 2014

Name: Key

Problem	Score	Possible
1		20
2		20
3		20
4		20
5		20
$\sum$		100

Rubric
5 pts. each
3-4 with poor reason
3 with no reason

1. Mark each statement below true or false. Give a short reason for each of your responses.

(a) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x^2 - y^2 = 1)$

FALSE: If $x=0$, then there is no real y satisfying the equation $(-y^2=1)$.

(b) $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}\ (x^2 - y^2 = 1)$

TRUE: If $y \in \mathbb{R}$, then $x = \sqrt{1+y^2}$ satisfies the equation.

(c) $p \leftrightarrow q$ is logically equivalent to $\neg p \leftrightarrow \neg q$

TRUE:
$$\begin{aligned} p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) \\ &\equiv (\neg q \to \neg p) \wedge (\neg p \to \neg q) \quad \text{[contrapositives]} \\ &\equiv \neg p \leftrightarrow \neg q \end{aligned}$$

(or: $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are each true iff. p, q have the same truth value)

(d) $[\exists x\ P(x)] \wedge [\exists x\ Q(x)]$ is logically equivalent to $\exists x\ [P(x) \wedge Q(x)]$

FALSE: As a counterexample, consider the predicates $P(x)$: "x is odd" and $Q(x)$: "x is even" with domain $\mathbb{Z}$. Then $[\exists x\ P(x)] \wedge [\exists x\ Q(x)]$ is true, but $\exists x\ [P(x) \wedge Q(x)]$ is false. (The first proposition says that there is an odd integer and there is an even integer. The second says that there is an integer that is both odd and even.)

2. Let A, B be sets. Show that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

We'll show that $(A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B)$
and that $(A-B) \cup (B-A) \supseteq (A \cup B) - (A \cap B)$.

($\subseteq$) Let $x \in (A-B) \cup (B-A)$.
Then either $x \in (A-B)$ or $x \in (B-A)$.
If $x \in (A-B)$, then $x \in A$ and $x \notin B$, so $x \in A \cup B$ but $x \notin A \cap B$.
If $x \in (B-A)$, then $x \in B$ and $x \notin A$, so $x \in A \cup B$ but $x \notin A \cap B$.
Either way, $x \in (A \cup B) - (A \cap B)$.

($\supseteq$) Let $x \in (A \cup B) - (A \cap B)$.
Then $x \in A \cup B$, so $x \in A$ or $x \in B$.
But $x \notin A \cap B$, so these cases are exclusive.
If $x \in A$, then $x \notin B$, so $x \in A - B$.
If $x \in B$, then $x \notin A$, so $x \in B - A$.
Either way, $x \in (A-B) \cup (B-A)$.

We conclude that $(A-B) \cup (B-A) = (A \cup B) - (A \cap B)$. $\square$

<u>Rubric</u>

20 pts. for the above argument OR correct use of Venn diagrams/membership tables OR de Morgan/distributive laws

10 pts. for just $\subseteq$ or just $\supseteq$

−2 for a minor error or unclear step

−4 for one de Morgan error

0 to 10 for a solution with major errors

3. Let x, y, z are integers, no two of which are equal.

   Show that if $x+y+z=13$, then $\max(x,y,z) \geq 6$.

We argue by contraposition.

Suppose $\max(x,y,z) < 6$. We may assume without loss of generality that $x > y > z$. Since x, y, z are distinct integers, we have $x \leq 5$, $y \leq 4$, $z \leq 3$, and so $x+y+z \leq 5+4+3 = 12$. In particular, $x+y+z \neq 13$.

Since $\max(x,y,z) < 6$ implies $x+y+z \neq 13$, $x+y+z=13$ implies $\max(x,y,z) \geq 6$. $\square$

Rubric

20 pts. for the above OR correct proof by cases

−2 for assuming x, y, z positive without a rigorous justification

−3 for an unclear step (minor)

−2 for one omission in a proof by exhaustion

10+ for exhaustion w/ several omissions

0 to 10 for a solution with major errors

No penalty for "$x \neq y \neq z$", but this is undesirable notation as it leaves doubt about whether $x \neq z$ is intended!

4. (a) Find an inverse of 7 modulo 100.

Rubric

10 pts. for (a)
5 for largely correct method w/ an error

$$100 = 14(7) + 2$$
$$7 = 3(2) + 1$$

$$\begin{aligned} 1 &= 7 - 3(2) \\ &= 7 - 3\big(100 - 14(7)\big) \\ &= 43(7) - 3(100) \end{aligned}$$

$\therefore\ 43(7) \equiv 1 \pmod{100}$

i.e. $\boxed{43}$ is an inverse of 7

(b) Consider the function $f : \{0, 1, 2, \ldots, 98, 99\} \to \{0, 1, 2, \ldots, 98, 99\}$ defined by $f(n) = 7n \,\underline{\text{mod}}\, 100$. For example, $f(20) = 40$.

Determine, with justification, whether f is injective and whether f is surjective.

YES to both.

Rubric

5 for inj., 5 for surj.
3[redacted] w/ poor reason or no reason

Proof that f is injective:

$$\begin{aligned} f(a) = f(b) &\implies 7a \equiv 7b \pmod{100} \\ &\implies 43(7a) \equiv 43(7b) \pmod{100} \\ &\implies a \equiv b \pmod{100} \\ &\implies a = b, \text{ since } 0 \le a, b \le 99. \end{aligned}$$

An injection between two <u>finite</u> sets of equal cardinality must be surjective.

(We can also show that f is bijective by writing down an explicit inverse:
$f^{-1}(n) = 43n \ \underline{\text{mod}}\ 100$.)

5. Determine the smallest **two** solutions of the system

$$\begin{cases} n \equiv 1 \pmod 5 \\ n \equiv 3 \pmod 7 \\ n \equiv 8 \pmod 9 \end{cases},$$

where $n \in \mathbb{Z}^+$.

Rubric

20 pts. for $n = 206, 521$ by any method

15 for $n = n_0,\ n_0 + 315$ w/ a minor error in working out n_0

≤ 10 for unfinished work/ only one n/ two n not differing by 315

≤ 5 with a major error, such as using the wrong definition of an inverse

Back-substitution:

$n \equiv 1 \pmod 5 \Rightarrow n = 5k+1$

$5k+1 \equiv 3 \pmod 7 \Rightarrow 5k \equiv 2 \pmod 7$

$\Rightarrow 3(5k) \equiv 3(2) \pmod 7$

$\Rightarrow k \equiv 6 \pmod 7$

$\Rightarrow k = 7\ell + 6$

$\Rightarrow n = 5(7\ell+6)+1 = 35\ell + 31$

$35\ell + 31 \equiv 8 \pmod 9 \Rightarrow 35\ell \equiv -23 \pmod 9$

$\Rightarrow 8\ell \equiv 4 \pmod 9$

$\Rightarrow 8(8\ell) \equiv 8(4) \pmod 9$

$\Rightarrow \ell \equiv 32 \equiv 5 \pmod 9$

$\Rightarrow \ell = 9m + 5$

$\Rightarrow n = 35(9m+5)+31 = 315m + 206$

The smallest two positive solutions are $\boxed{n = 206,\ 521}$.

Tabular method: $M = 5 \cdot 7 \cdot 9 = 315$

a_i	m_i	M_i	y_i
1	5	$7 \cdot 9 = 63$	2
3	7	$5 \cdot 9 = 45$	5
8	9	$5 \cdot 7 = 35$	8

Efficient computation: $M_1 y_1 \equiv 1 \pmod 5 \Leftrightarrow 3y_1 \equiv 1 \pmod 5$; use trial and error with $y_1 \in \{0,1,2,3,4\}$.

$n = \sum_{i=1}^{3} a_i M_i y_i = 3041$ is one solution.

All solutions: $n \equiv 3041 \pmod{315}$

Smallest positive solutions:

$n = 206, 521$

✱ REMARK: People had much higher success rates with back-substitution. ✱