Math 55: Review Problems for Midterm #2

Summer 2014

Problems with one asterisk are on the harder side. The problem with two asterisks is quite hard and would never appear on a Math 55 exam.

Combinatorial proof

In problems 1–11, prove the identity by a counting argument. The first three are standard theorems.

1. $\binom{n}{k} = \binom{n}{n-k}$
2. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
3. $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
4. $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$
5. $\binom{2n}{2} = n^2 + 2\binom{n}{2}$
6. $\binom{3n}{2} = 3n^2 + 3\binom{n}{2}$
7. $\binom{2n}{3} = 2\binom{n}{3} + 2n\binom{n}{2}$
8. $\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2} = \frac{(2n)!}{2!^n}$
9. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$
10. $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$

* 11. $\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+k}{n} = \binom{n+k+1}{n+1}$

Counting and probability

12. How many 5-letter strings have exactly 4 different letters (like PANDA or GIMME)?
13. How many positive integers have their digits in strictly increasing order (like 25679)?
14. In how many ways can we choose a subset of an n-element set and distribute the elements of the subset to k boxes? The elements are distinguishable and so are the boxes.
15. There are two decks of cards. One is a standard 52-card deck; the other is a 54-card deck with two jokers. We don't know which deck is which, so we pick a deck at random and draw a card. If that card is not a joker, what's the probability that we drew from the 52-card deck?
16. How many 3-digit numbers must we choose to ensure that some two of the chosen numbers have the same digits, not necessarily in the same order (like 144 and 441)?

* 17. Two 5-card poker hands are dealt from the same deck (without replacement). Each hand turns out to be a flush (5 cards of the same suit; for this problem, we'll consider a straight flush to be a kind of flush). What is the probability that both flushes are in the same suit?

* 18. Here are two possible methods of distributing 6 indistinguishable balls to 3 distinguishable boxes so that no box is empty:

- Method 1: Start with 1 ball in each box. For each remaining ball, choose a box uniformly at random and put the ball in that box.
- Method 2: For each ball, choose a box uniformly at random and put the ball in that box. After placing all 6 balls in this way, if any box is empty, throw out the results and start over. Repeat until you obtain a result with at least one ball in each box.

Do these two methods lead to the same distribution of outcomes?

Expected value

19. We choose an integer from 1 to 9999 uniformly at random. What is the expected number of digits?

20. We choose a subset of $\{1, 2, 3, \ldots, n\}$ uniformly at random. What is the expected cardinality?

* 21. A fair coin is flipped repeatedly. What is the expected number of flips if we stop when: (a) a head and a tail appear consecutively, in that order? (b) two heads appear in a row?

* 22. Suppose you start with \$1. You repeatedly flip a fair coin; on heads you win \$1, and on tails you lose \$1. If you lose all your money, you have to stop playing. What is the probability that you ever reach \$20?

** 23. If a monkey types a random letter of the English alphabet every second, how long will it take, on average, for the word ABRACADABRA to appear?

Induction and recursion

24. Suppose we can move from (x, y) to $(x+1, y)$, $(x, y+1)$, or $(x+1, y+1)$ in a single step. How many sequences of steps are there... (a) from $(0, 0)$ to $(n, 1)$? (b) from $(0, 0)$ to $(3, 3)$?

25. Let a_n be the number of n-letter strings with no two consecutive vowels (A, E, I, O, U). Write a recurrence relation and initial conditions for $\{a_n\}$.

26. Let m be an odd positive integer. Prove from basic principles (i.e. don't cite the division algorithm) that for every integer n, there is some integer r such that $|r| \leq \lfloor m/2 \rfloor$ and $n \equiv r \pmod{m}$.

* 27. Prove Fermat's Little Theorem by induction. Hint: Most of the terms in the expansion of $(a+1)^p$ are divisible by p (why?).