Day 21 (Wed 7/23)

$\boxed{\text{Opened with a discussion of lexicographic order.}}$

Hour 1

Problem: How many ways can 6 indistinguishable objects be placed into 3 indistinguishable boxes:
— Answer: 7 ways: (6,0,0), (5,1,0), (4,2,0), (4,1,1), (3,3,0), (3,2,1), (2,2,2).

Compare to the 28 ways possible if the boxes are distinguishable. (Why 28?)

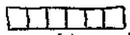
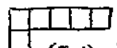
$$\left.\begin{matrix}(6,0,0)\\(0,6,0)\\(0,0,6)\end{matrix}\right\}\rightarrow(6,0,0) \qquad \left.\begin{matrix}(5,1,0) & (1,0,5)\\(5,0,1) & (0,5,1)\\(1,5,0) & (0,1,5)\end{matrix}\right\}\rightarrow(5,1,0) \quad \text{etc.}$$

[Avoid repetition by requiring $x_1 \geq x_2 \geq x_3$.]

Since these groupings are not of equal size, we can't apply the division rule. Thus we have a new counting problem...

Def'n.) Let n, k be ~~positive~~ [nonnegative] integers. A <u>partition</u> of n into k parts is a k-tuple of positive integers $(n_1, n_2, \ldots, n_k)$ such that $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_k$ and $n_1 + n_2 + \cdots + n_k = n$.

Partitions of 6 into...
... 1 part: (6)
... 2 parts: (5,1), (4,2), (3,3)
[etc., up to 6 parts]

Young diagrams:  (6),  (5,1), etc. (Characterize in terms of closure under going up or left.)

Theorem: The number of partitions of n into exactly k parts is equal to the number of partitions of n whose largest part is k.

Proof: The former partitions are the ones whose Young diagrams have k rows; the latter, k columns. These are in bijective correspondence (by reflection across the diagonal).

Def'n.) Let $\text{par}(n,k)$ be the number of partitions of n into exactly k parts. By definition,
Ex: $\text{par}(6,1) = 1$; $\text{par}(6,2) = 3$; $\text{par}(6,3) = 3$; $\text{par}(6,4) = 2$; $\text{par}(6,5) = 1$; $\text{par}(6,6) = 1$.

Theorem: The numbers $\text{par}(n,k)$ are given recursively by these rules:
- $\text{par}(0,0) = 1$
- $\text{par}(n,0) = 0$ for $n \geq 1$
- $\text{par}(n,k) = 0$ for $k > n$
- $\text{par}(n,k) = \text{par}(n-1,k-1) + \text{par}(n-k,k)$ for $1 \leq k \leq n$.

Hour 2

Prove the theorem. Main step (recursive rule): Each partition of n into k parts either has smallest part 1 or has every part ≥ 2. Partitions of the first kind, $(n_1, n_2, \ldots, n_{k-1}, 1)$, are in bijection with partitions of $n-1$ into $k-1$ parts: $(n_1, n_2, \ldots, n_{k-1})$. Partitions of the second kind, $(n_1, n_2, \ldots, n_k)$, are in bijection with partitions of $n-k$ into k parts: $(n_1 - 1, n_2 - 1, \ldots, n_k - 1)$ (note that $n_k - 1 \geq 1$).

Use the theorem to tabulate $\text{par}(n,k)$ and total partitions for $n \leq 7$.

Extremely optional: Show that the number of partitions of n with diagonal symmetry is equal to the number of partitions of n whose parts are odd and distinct. (e.g., for $n = 16$...) Or discuss distinguishable objects in indistinguishable boxes.

Something completely different:

Alternative proof of Fermat's Little Theorem:

Lemma: If p is prime and $1 \leq k \leq p-1$, then p divides $\binom{p}{k}$. [Demonstrate for $p = 7$.]
—Solicit proof.

Theorem: If p is prime and $a \geq 1$ is an integer, then $a^p \equiv a \pmod{p}$.

Proof: Induction on a. Basis step: $1^p \equiv 1 \pmod{p}$.
Inductive step: If $a^p \equiv a \pmod{p}$, then

$$(a+1)^p = a^p + \binom{p}{1}a^{p-1} + \binom{p}{2}a^{p-2} + \cdots + \binom{p}{p-1}a + 1$$
$$\equiv a + 0 + 0 + \cdots + 0 + 1 \pmod{p}.$$

not covered