1. Suppose we have an unlimited supply of 4-cent and 15-cent stamps.

   (a) Prove that we can't make 41 cents postage.

   Suppose we can make 41¢ from s 4¢ stamps and t 15¢ stamps.
   Then $41 = 4s + 15t$ (where $s,t \in \mathbb{N}$).

   If $t=0$, then $s = \frac{41}{4} \notin \mathbb{N}$, a contradiction.

   If $t=1$, then $s = \frac{26}{4} \notin \mathbb{N}$, " "

   If $t=2$, then $s = \frac{11}{4} \notin \mathbb{N}$, " "

   If $t \geq 3$, then $4s+15t \geq 45 > 41$, also a contradiction.

   Thus we cannot make 41¢.

   (b) Prove that we can make n cents postage for all integers $n \geq 42$.

   We argue by strong induction.
   We can make $n = 42, 43, 44, 45$ as follows:

   $$42 = 3(4) + 2(15)$$
   $$43 = 7(4) + 1(15)$$
   $$44 = 11(4) + 0(15)$$
   $$45 = 0(4) + 3(15)$$

   Suppose we can make $n = 42, 43, \ldots, k-1$, where $k \geq 46$.
   Then, in particular, we can make $k-4$ cents postage. By adding a 4-cent stamp, we can then make k cents.

   We conclude by strong induction that we can make n cents postage for all integers $n \geq 42$.

2. 60% of all e-mail Mira receives is spam. If a message is spam, there's a 3% probability that it contains the word "opportunity". If a message is not spam, there's a 0.5% probability that it contains the word "opportunity".

   What is the probability that a message containing the word "opportunity" is spam?

Let E be the event "is spam". Let O be the event "contains 'opportunity'". Then we are given that $p(E) = 0.6$, $p(O|E) = 0.03$, and $p(O|\overline{E}) = 0.005$. The probability we are seeking is $p(E|O)$.

By Bayes' Theorem,

$$p(E|O) = \frac{p(E)p(O|E)}{p(E)p(O|E) + p(\overline{E})p(O|\overline{E})} = \frac{(0.6)(0.03)}{(0.6)(0.03)+(0.4)(0.005)} = \frac{0.018}{0.018+0.002}$$

$$= \boxed{\frac{9}{10}} \text{ or } \boxed{0.9} \text{ or } \boxed{90\%}$$

3. In a standard deck of 52 cards, there are 13 *ranks* (labeled 2, 3, …, 10, J, Q, K, A) and 4 *suits* (♣, ♢, ♡, ♠). There is one card of each rank-suit combination (e.g., Q♡ is a card). A hand is a subset of the deck; the order of cards in a hand does not matter.

   (a) How many 5-card hands are there in which no rank is repeated?

   There are $\binom{13}{5}$ ways to choose 5 distinct ranks and 4^5 ways to choose the suit associated with each rank, for

   $$\boxed{\binom{13}{5} \cdot 4^5} = \frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5!}$$ possible hands.

   (b) In total, how many 1- to 13-card hands are there in which no rank is repeated? Express your answer in as simple a form as possible.

   Method 1

   By the reasoning in (a), there are $\binom{13}{k} \cdot 4^k$ k-card hands with no repeated ranks. Summing over all k from 1 to 13 yields $\sum_{k=1}^{13} \binom{13}{k} \cdot 4^k$ hands; by the binomial theorem, this sum equals $\boxed{5^{13} - 1}$.

   Method 2

   For each rank, we can choose one of 5 options: include the club, diamond, heart, spade, or no card of that rank. Thus we can form $\underbrace{5 \times 5 \times 5 \times \cdots \times 5}_{13\ 5\text{'s}}$ hands, one of which is the empty hand. The number of 1- to 13-card hands is $\boxed{5^{13} - 1}$.

(Continued.)

(c) What is the probability that a random 5-card hand contains at least one heart (♡)?

All $\binom{52}{5}$ hands are equally probable; $\binom{39}{5}$ of them have no hearts, so the probability of getting at least one heart is

$$\boxed{1-\frac{\binom{39}{5}}{\binom{52}{5}}}.$$

(d) What is the expected number of different suits in a random 5-card hand?

Let $X_1 = \begin{cases} 1 & \text{if we draw at least one } \heartsuit \\ 0 & \text{otherwise} \end{cases}$.

Let X_2, X_3, X_4 be similarly defined for the other suits.

Then $E(X_1) = 1-\frac{\binom{39}{5}}{\binom{52}{5}}$ by part (c), and $E(X_2)$, $E(X_3)$, $E(X_4)$ are the same.

If X is the number of different suits drawn, then $X = X_1+X_2+X_3+X_4$.

By linearity of expectation, $E(X) = E(X_1)+E(X_2)+E(X_3)+E(X_4)$

$$= \boxed{4\left(1-\frac{\binom{39}{5}}{\binom{52}{5}}\right)}.$$

4. Prove the identity $k\binom{n}{2} + \binom{k}{2}n^2 = \binom{kn}{2}\ldots$

(a) … algebraically.

$$\begin{aligned} k\binom{n}{2} + \binom{k}{2}n^2 &= k\cdot\frac{n(n-1)}{2} + \frac{k(k-1)}{2}\cdot n^2 \\ &= \frac{kn\,[n-1+(k-1)\cdot n]}{2} \\ &= \frac{kn\,[n-1+kn-n]}{2} \\ &= \frac{kn\,(kn-1)}{2} \\ &= \binom{kn}{2} \end{aligned}$$

(b) … by a combinatorial argument.

Suppose we have kn balls of k colors, with n in each color. Then there are $k\binom{n}{2}$ ways to choose two balls of the same color (because we can choose the color in k ways and the balls of that color in $\binom{n}{2}$ ways).

There are $\binom{k}{2}n^2$ ways to choose two balls of different colors (because we can choose the two colors in $\binom{k}{2}$ ways, the ball of the first color in n ways, and the ball of the second color in n ways).

Two balls are either the same color or different colors, so we have a total of $k\binom{n}{2}+\binom{k}{2}n^2$ ways to choose any two balls — but we also have $\binom{kn}{2}$ ways to do that, by definition. So $k\binom{n}{2}+\binom{k}{2}n^2=\binom{kn}{2}$.

5. For integers $n \geq 1$, let a_n be the number of subsets of $\{1, 2, 3, \ldots, n\}$ that do not contain two consecutive integers. (Include the empty set.)

Determine sufficient initial values and a recurrence relation for the sequence $\{a_n\}_{n=1}^{\infty}$. Justify your answer.

Let us call a subset of $\{1,2,3,\ldots,n\}$ <u>good</u> if it does not contain two consecutive integers.

Then a good subset of $\{1,2,3,\ldots,n\}$ either does or does not contain n. If it does contain n, then it cannot contain $n-1$; such sets are of the form $A \cup \{n\}$, where A is a good subset of $\{1,2,3,\ldots,n-2\}$. (assuming $n \geq 3$)

If it does not contain n, then it is a good subset of $\{1,2,3,\ldots,n-1\}$. There are a_{n-2} good subsets of $\{1,2,3,\ldots,n\}$ that contain n and [redacted] a_{n-1} good subsets of $\{1,2,3,\ldots,n\}$ that do not contain n. Thus $a_n = a_{n-2} + a_{n-1}$ for $n \geq$ [redacted]3.

The recurrence is of order 2, so we need two initial values.
The good subsets of $\{1\}$ are $\emptyset$ and $\{1\}$, so $a_1 = 2$.
The good subsets of $\{1,2\}$ are $\emptyset, \{1\}$, and $\{2\}$, so $a_2 = 3$.

Remark: a_n is the $(n+2)^{\text{th}}$ Fibonacci number, but stating this was not required.

Rubric

#1 10 pts. each part.

On (a): 9 pts. for exhaustion/cases with no justification for ≤ 2 15¢ stamps.
7 pts. for exhaustion/cases with missing cases.
≤ 5 pts. for unsupported claim of exhaustion.

On (b): Valid strong induction, weak induction, and non-inductive proofs were seen.
5 to 8 pts. for hand-wavey style or inappropriate presentation (e.g. undefined "$P(n)$", misstated inductive hypothesis, misuse of variables)
5 pts. for strong induction with only one base case (if not sufficient)
5 pts. for "turn 15¢ into 4 × 4¢" arguments that ignore possibility of no 15¢ stamps
2 pts. for just base case(s)

#2 20 pts.
17 pts. for correct setup w/ arithmetic error or misplaced decimal point
10 pts. for correct setup w/ some probabilities wrong (other than misplaced decimal)
≤ 5 pts. for incorrect setup of Bayes or for finding wrong probability, e.g. $p(\text{spam} \wedge \text{"opp."})$

#3 5 pts. each part; generous PC as follows:

(a) 3 pts. for counting ordered hands, i.e. $P(13,5)\cdot 4^5$
2 pts. for choosing only one suit, i.e. $4\cdot\binom{13}{5}$
No credit for both mistakes

(b) 4 pts. for $4^k\binom{13}{k}$ or $\sum_{k=1}^{13} 4^k\binom{13}{k}$ (unsimplified)
4 pts. for 5^{13} (includes empty hand)
3/2 pts. for mistakes analogous to those in (a)

(c) 2 pts. for treating cards as independent/drawn with replacement, i.e. $1-\left(\frac{3}{4}\right)^5$
2 pts. for $\dfrac{\binom{13}{1}\binom{51}{4}}{\binom{52}{5}}$ (some hands are overcounted)

(d) 5 pts. for correct answer even if done the long, painful way (i.e. $p(X=1)+2p(X=2)+3p(X=3)+4p(X=4)$)
4 pts. for long, painful answer with a mistake in one case
2 pts. for $E(X)=p(X=1)+2p(X=2)+3p(X=3)+4p(X=4)$ with poor attempt to compute probabilities
No credit for claiming # of suits is geometrically distributed or for $E(X)=5\cdot\frac{1}{4}$

#4 10 pts. each part.
On (a): 9 pts. if steps are logically reversed.
On (b): Range of scores for proofs lacking clarity/detail.

#5 20 pts.
17 pts. for listing 1- and 2-element sets but not stating $a_1=2$, $a_2=3$.
15 pts. for correct result with poor justification.
10 pts. for correct result with no justification other than apparent pattern for small n.
≤ 5 pts. for serious misunderstanding of problem.