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1. A true/false test has 50 questions, each worth 2 points. Carmen has a 90% chance of answering each question correctly (and her performance on each question is independent from her performance on other questions).

   What are the expected value and variance of Carmen's total score?

Let X_i be Carmen's score for the i^{th} question (which is 2 or 0).

Then $E(X_i) = (0.9)(2) + (0.1)(0) = 1.8$ and

$V(X_i) = (0.9)((2-1.8)^2) + (0.1)((0-1.8)^2) = 0.36.$ (for $1 \le i \le 50$)

Let X be Carmen's total score; $X = \sum_{i=1}^{50} X_i$.

By linearity of expectation, $E(X) = \sum_{i=1}^{50} E(X_i) = 50(1.8) = \boxed{90}$.

Since $X_1, X_2, \ldots, X_{50}$ are independent, we also have

$V(X) = \sum_{i=1}^{50} V(X_i) = 50(0.36) = \boxed{18}$.

2. If $a_1 = 6$, $a_2 = 12$, and $a_n = 4a_{n-1} - 3a_{n-2}$ for $n \geq 3$, solve for a_n in closed form.

Characteristic polynomial is $r^2 - 4r + 3 = 0$, with roots $r = 1, 3$.
Thus the solution is of the form $a_n = s(1)^n + t(3)^n$.

Substitute $n = 1, 2$ to get the system

$$\begin{cases} 6 = s + 3t \\ 12 = s + 9t \end{cases}$$

whose solution is $t = 1$, $s = 3$. Thus $a_n = 3(1)^n + 1(3)^n = \boxed{3^n + 3}$ $(n \geq 1)$.

3. Let $f : A \to B$ and $g : B \to A$ be functions such that $g(f(x)) = x$ for all $x \in A$ and $f(g(x)) = x$ for all $x \in B$. Show that f must be a bijection.

To show f is injective, we must show that $f(x) = f(y) \Rightarrow x = y$.

If $f(x) = f(y)$, then $g(f(x)) = g(f(y))$, so $x = y$.
Thus f is injective.

To show f is surjective, we must show that, for every $x \in B$, there is some $y \in A$ such that $f(y) = x$. There is: namely, if $y = g(x)$, then $f(y) = f(g(x)) = x$. Thus f is surjective.

Since f is injective and surjective, f is a bijection.

4. Each deck of cards in this problem is a well-shuffled standard 52-card deck with 13 hearts. Alice draws 5 cards from 1 deck. Bob draws 1 card from each of 5 decks.

   (a) Who has greater probability of drawing 5 hearts? Check one box:

   ☐ Alice ☑ Bob ☐ Equal

   Show your work here:

   Alice's probability is $\dfrac{\binom{13}{5}}{\binom{52}{5}} = \dfrac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$.

   Bob's probability is $\left(\dfrac{13}{52}\right)^5 = \left(\dfrac{1}{4}\right)^5$.

   Since $\dfrac{13-k}{52-k} < \dfrac{1}{4}$ for $k=1,2,3,4$, we have $\dfrac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} < \left(\dfrac{1}{4}\right)^5$.

   (b) Who draws a greater expected number of hearts? Check one box:

   ☐ Alice ☐ Bob ☑ Equal

   Show your work here:

   Each card drawn has probability $\frac{1}{4}$ of being a heart.
   By linearity of expectation (applied to indicator random variables for the event of each card being a heart), Alice and Bob each have an expectation of $\frac{5}{4}$ hearts.

5. Let a be an integer such that $\gcd(a, 55) = 1$. Show that $a^{20} \equiv 1 \pmod{55}$.

If $\gcd(a,55) = 1$, then $5 \nmid a$ and $11 \nmid a$.

By Fermat's Little Theorem, $a^4 \equiv 1 \pmod{5}$
and $a^{10} \equiv 1 \pmod{11}$.

Thus $a^{20} \equiv (a^4)^5 \equiv 1^5 \equiv 1 \pmod{5}$
and $a^{20} \equiv (a^{10})^2 \equiv 1^2 \equiv 1 \pmod{11}$.

Since $5 \mid a^{20}-1$ and $11 \mid a^{20}-1$, it follows that $55 \mid a^{20}-1$
and therefore $a^{20} \equiv 1 \pmod{55}$.

6. Let $m \leq n$ and let $K_{m,n}$ denote the complete bipartite graph on sets of m and n vertices.

(a) How many distinct vertices must be chosen from $K_{m,n}$ to ensure that some two adjacent vertices are chosen?

It suffices to choose $\boxed{n+1}$ vertices, since they can't all be on the same side of the vertex bipartition.

(b) How many distinct vertices must be chosen from $K_{m,n}$ to ensure that some two non-adjacent vertices are chosen?

It suffices to choose $\boxed{3}$ vertices, since (by Pigeonhole) two will be on the same side of the bipartition.

7. Determine the coefficient of x^{10} in the expansion of $(x + x^2 + x^3 + \cdots)^6$.

Each term in the expansion is of the form $x^{a_1}x^{a_2}x^{a_3}x^{a_4}x^{a_5}x^{a_6}$, where $a_1, \ldots, a_6$ are integers such that $a_i \geq 1$ for $i = 1, \ldots, 6$. Thus the coefficient of x^{10} is the number of solutions to $a_1 + a_2 + \cdots + a_6 = 10$ with $a_i \geq 1$. This is the same as the number of solutions to $a_1' + a_2' + \cdots + a_6' = 4$ with $a_i' \geq 0$ (via the substitution $a_i' = a_i - 1$).

The number of solutions is $\binom{4+6-1}{6-1} = \binom{9}{5} = \boxed{126}$.

8. Let R be the relation on $\mathbb{N}$ such that $m\ R\ n$ if and only if 3 divides $2^m - 2^n$.

(a) Show that R is an equivalence relation.

R is reflexive because $3 \mid 2^n - 2^n$ $(= 3 \cdot 0)$ for all $n \in \mathbb{N}$.

R is symmetric because $m R n \Rightarrow 3 \mid 2^m - 2^n \Rightarrow$
$3 \mid (-1)(2^m - 2^n) \Rightarrow 3 \mid 2^n - 2^m \Rightarrow n R m$.

R is transitive because if $m R n$ and $n R k$, then $3 \mid 2^m - 2^n$
and $3 \mid 2^n - 2^k$, which implies $3 \mid (2^m - 2^n) + (2^n - 2^k)$,
which implies $3 \mid 2^m - 2^k$; so $m R k$.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation.

(b) What are the equivalence classes?

If n is even, then $2^n \equiv (-1)^n \equiv 1 \pmod 3$.
If n is odd, then $2^n \equiv (-1)^n \equiv -1 \pmod 3$.
Thus the equivalence class are $\{0, 2, 4, 6, \ldots\}$ and $\{1, 3, 5, 7, \ldots\}$.

9. How many anagrams of HEDGEHOG begin and end with the same letter?
(An anagram is an arrangement of all the letters. It does not need to be an English word.)

There are three cases: H——H, E——E, and G——G.

In the case H——H, the remaining letters are 2 E's, 2 G's, 1 D, and 1 O, which can be arranged in $\binom{6}{2,2,1,1} = \frac{6!}{2!2!}$ ways.

The other cases are like the first one.

Thus the total number of anagrams is $3 \cdot \frac{6!}{2!2!} = \boxed{540}$.

10. Let G be an undirected graph with n vertices and m edges, where $1 \leq m \leq n-1$. Suppose G is connected. Show that G has a vertex of degree 1.

We argue by contradiction. Suppose G has no vertex of degree 1. Since G is connected and has at least one edge, G cannot have a vertex of degree 0. Thus every vertex has degree ≥ 2.

The total degree of all vertices is at least $2n$. By the Handshaking Theorem, the total degree of all vertices equals $2m$. Thus $2m \geq 2n$, so $m \geq n$. But it was given that $m \leq n-1$, so we have a contradiction.

We conclude that G must have a vertex of degree 1.