Math 55: Final Review Problems

Summer 2014

Chapter 1: Logic

1. In this problem, the domain for all predicates is the set of people. If $F(p_1, p_2)$ is the predicate "p_1 is p_2's father", and $M(p_1, p_2)$ is the predicate "p_1 is p_2's mother", use logical connectors to write a predicate that asserts "p_1 is p_2's grandmother".
2. Let p, q, r be propositions. Given the premises $p \to r$, $q \to r$, and $p \vee q$, show that we can validly infer r. (What method of proof is based on this rule of inference?)
3. Let $P(1), P(2), P(3), \ldots$ be a sequence of propositions. Given the premises $P(1)$ and $\forall\ k \geq 1\ [P(k) \to P(k+1)]$, show that we can validly infer $\forall\ n \geq 1\ P(n)$.

Chapter 2: Sets and Functions

4. Let A and B be sets such that $|A| = 5$, $|B| = 4$, and $|A \cap B| = 2$. Determine $|\mathcal{P}(A) \cup \mathcal{P}(B)|$, where $\mathcal{P}(S)$ denotes the set of all subsets of S.
5. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function given by $f(x, y) = xy + x + y$. Determine whether f is an injection and whether f is a surjection.
6. Show, by an example, how a set can have the same cardinality as a proper subset of itself.

* 7. Let S be a countable set. Let $\mathcal{P}^*(S)$ be the set of **finite** subsets of S. What is the cardinality of $\mathcal{P}^*(S)$? Why?

Chapter 4: Primes and Modular Arithmetic

8. Let m be a positive integer. Working from the definition of modular congruence, show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.
9. Let a, b, q, r be integers such that $a = qb + r$. Show that $\gcd(a, b) = \gcd(b, r)$. This result is used to prove the validity of what important procedure?
10. Determine a positive integer value of n such that $2^n \equiv 1 \pmod{143}$.

Chapter 5: Induction and Recursion

11. Show that every integer greater than or equal to 2 can be written as a product of primes.
12. The Fibonacci numbers are defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. The Lucas numbers are defined by $\ell_1 = 1$, $\ell_2 = 3$, and $\ell_n = \ell_{n-1} + \ell_{n-2}$ for $n \geq 3$. Show that $\ell_n = f_{n-1} + f_{n+1}$ for all integers $n \geq 2$.

Chapter 6: Counting Basics

13. Count:
    (a) functions from an n-element set to a k-element set
    (b) injective functions from an n-element set to a k-element set
    (c) diagonals in an n-sided regular polygon
    (d) ways to travel from $(0, 0)$ to (m, n) by taking steps of 1 unit to the right or 1 unit up
14. How many integers must be chosen to ensure that there are some two chosen integers whose difference is a multiple of 10?
15. How many integers must be chosen to ensure that there are some two chosen integers whose difference or sum is a multiple of 10?
16. State Pascal's identity and prove it by a counting argument.
17. Count:
    (a) ways to choose sizes for 6 T-shirts where the possible sizes are S, M, L, and XL, and the order of the shirts does not matter
    (b) ways to deal out four 5-card hands from a 52-card deck, if it matters who gets which cards but not what order they get them in
    (c) anagrams of BANANAS that don't start with N
    (d) anagrams of CHEESE in which no two E's are in consecutive positions

Chapter 7: Discrete Probability

18. On a true/false test with 10 questions, Cody has an 80% chance of answering each question correctly (and her performance on one question is independent from her performance on the other questions). What is the probability that Cody's actual score is equal to her expected score?
19. A coin comes up heads with probability p on any given flip. If the coin is flipped n times, what is the expected value and variance of the number of heads?
20. If it rains on a given day, there is a 60% chance it will rain the next day. If it doesn't rain on a given day, there is a 20% chance it will rain the next day. Given these probabilities:
    (a) If it rained two days ago, what is the probability that it rains today?
    (b) If it rained two days ago and it rains today, what is the probability that it rained yesterday?
21. A 3×3 grid is randomly filled with 0's and 1's such that each entry is 0 or 1 with probability 1/2, independent of the other entries. What is the expected number of positions where two horizontally or vertically adjacent entries are both 0's?

Chapter 8: Advanced Counting

22. Count solutions to $x_1 + x_2 + x_3 + x_4 = 10$ where each x_i is an integer from 0 to 4

23. Let a_n be the number of ways to make change for n cents using pennies, nickels, and dimes. Write down the generating function for $\{a_n\}_{n=0}^{\infty}$ (you do not need to expand it out).

24. What is the coefficient of x^n in the expansion of $(1 + x + x^2 + x^3 + \cdots)^k$?

25. Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$, where $a_1 = 1$ and $a_2 = 3$.

Chapters 9–10: Relations and Graphs

26. Let R be a relation on the set of all integers, such that $m\ R\ n$ if and only if $m^2 - n^2$ is divisible by 5. Show that R is an equivalence relation. Determine the equivalence classes.

27. Let G be a connected, undirected simple graph on n vertices. Let R be a relation on the vertices of G, such that $v\ R\ v'$ if and only if v is adjacent to v'. Show that if R is transitive, then $G = K_n$.

28. The graph Q_n has vertices labeled by n-bit strings (with exactly one vertex for each such string). Two vertices are adjacent in Q_n if and only if their corresponding strings differ in exactly one position.

    (a) How many vertices and edges does Q_n have?

    (b) Let u, v be two vertices of Q_n such that the distance from u to v is equal to the diameter of Q_n. How many paths of minimum length are there from u to v?

29. Show that an undirected graph is bipartite if and only if it has no circuits of odd length.

30. Let G, G' be simple graphs, each with n vertices and $\binom{n}{2} - 1$ edges. Show that G and G' must be isomorphic.