# Math 55: Discrete Mathematics, Fall 2012
# Final Exam

1. (8 pts) Solve the simultaneous congruences

$$x \equiv 3 \pmod 8$$
$$x \equiv 5 \pmod 9$$

2. (4 pts each) For each of the following give a yes or no answer and a one-sentence justification.

(a) Does an algorithm $A$ exist which takes as input a program $P$ and additional data $I$, and halts if and only if $P$ halts when run with input $I$?

(b) Does an algorithm $B$ exist which takes as input a program $P$ and additional data $I$, and halts if and only if $P$ *does not* halt when run with input $I$?

3. (5 right 8 pts, 4 right 5 pts, 3 right 2 pts) Let $S$ be the set of all finite subsets of $\mathbb{Z}$. Mark each of the following True or False. Here $\mathbb{Z}$ denotes the set of all integers, and $\mathcal{P}(A)$ denotes the power set of $A$.

(a) $\mathbb{Z} \in S$

(b) $\mathbb{Z} \subseteq S$

(c) $S \subseteq \mathcal{P}(\mathbb{Z})$

(d) $\mathcal{P}(\mathbb{Z}) \subseteq S$

(e) $S \cap \mathbb{Z} = \emptyset$

4. (8 pts) Do there exist irrational numbers $x$ and $y$ such that $x + y$ is rational? Prove that your answer is correct.

5. (8 pts) Which of the following is the most reasonable estimate for the number of steps Pollard's algorithm might take to factor the number $n = 72916997$: 10 steps, 100 steps, 1000 steps, 10,000 steps, 100,000 steps, or 1,000,000 steps?

Justify your choice. You may assume that the given number $n$ is a product of two 4-digit primes.

6. (5 pts each) Find the number of ways to assign 75 students to 3 discussion sections

(a) if each section must have 25 students;

(b) if there can be any number of students (including zero) in each section.

7. (8 pts) What is the largest number of elements that a set of integers can have if it does not contain three distinct elements $a$, $b$, $c$ such that $a \equiv b \equiv c \pmod{10}$?

8. (8 pts) Evaluate

$$\binom{7}{0} - 2\binom{7}{1} + 2^2\binom{7}{2} - 2^3\binom{7}{3} + 2^4\binom{7}{4} - 2^5\binom{7}{5} + 2^6\binom{7}{6} - 2^7\binom{7}{7}.$$

9. (8 pts) January 1, 2011 was a Saturday. Find the probability that a baby born in 2011 was born on Sunday, on Monday, and so on for each of the seven days of the week. Assume that the baby's birthday is equally likely to be any of the 365 days in the year.

10. (5 pts each) Let $X$ be the number that comes up when a fair 6-sided die is rolled once. We calculated $EX = 7/2$ and $V(X) = 35/12$ in class.

(a) Let $Y$ be the total when the die is rolled 12 times. Find $EY$ and $V(Y)$.

(b) Use the values of $EY$ and $V(Y)$ to find a lower bound on the probability that $33 \leq Y \leq 51$.

11. (8 pts) Find the probability that a number chosen at random between 1 and 100 is divisible by 5 or 7.

12. (8 pts) There are 5 waiters in a restaurant. One of them is color-blind. If you order tomato soup from the color-blind waiter there is a 50 percent chance that he will bring pea soup instead. You order tomato soup and receive the right soup. What is the probability that your waiter is the color-blind one?

Assume that the waiter for your table was chosen at random, and that only the color-blind waiter might bring the wrong soup.