Math 55: Discrete Mathematics
Solutions for the Final Exam

UC Berkeley, Spring 2012

1. (a) There are 3^n functions from $\{1, \ldots, n\}$ to $\{1, 2, 3\}$.

   (b) If $n \leq 3$ there are $P(3, n)$ injective functions. Hence, there are 3 when $n = 1$, 6 when $n = 2$ and 6 when $n = 3$. If $n > 3$, then there are 0 injective functions; there cannot be a 1-1 function from A to B if the cardinality of A is greater than the cardinality of B.

   (c) By inclusion-exclusion, the answer is

$$\begin{aligned}
&|\{f : \mathrm{ran}(f) \subseteq \{1,2,3\}\}| \\
&-|\{f{:}\mathrm{ran}(f) \subseteq \{1,2\}\}| - |\{f{:}\mathrm{ran}(f) \subseteq \{1,3\}\}| - |\{f{:}\mathrm{ran}(f) \subseteq \{2,3\}\}| \\
&+|\{f : \mathrm{ran}(f) \subseteq \{1\}\}| + |\{f : \mathrm{ran}(f) \subseteq \{2\}\}| + |\{f : \mathrm{ran}(f) \subseteq \{3\}\}| \\
&\qquad = 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 = 3^n - 3 \cdot 2^n + 3
\end{aligned}$$

2. Let a and b be any two vertices of G. If a and b are in different connected components of G, then there must be an edge from a to b in $\overline{G}$. If a and b are in the same connected component of G, then there must be a vertex c that is in a different connected component of G from a and b, and hence in $\overline{G}$ there path from a to b via c.

3. There are 8 possible outcomes in this experiment, all of them equally likely. Let X be the random variable that counts the number of edges that have both endpoints of the same color. By inspection we find that no outcome satisfies $X = 0$, six of the outcomes satisfy $X = 1$, and two of them satisfy $X = 3$. The expected value of X is then

$$E(X) = \frac{0}{8} \cdot 0 + \frac{6}{8} \cdot 1 + \frac{2}{8} \cdot 3 = \frac{12}{8} = \frac{3}{2}.$$

4. We fix the ground set $S = \{a,b,c,d\}$, and we consider the relation $R = \{(a,b),(b,c),(c,d)\}$. Then the transitive closure of R equals $R^* = \{(a,b),(b,c),(c,d),(a,c),(b,d),(a,d)\}$. On the other hand, $R^2 = \{(a,c),(b,d)\}$, and $R^3 = \{(a,d)\}$. Hence R^3 is necessary to get R^*.

5. We will prove this claim by induction. For the base case take $n = 1$. Note that
$$f_0 f_1 + f_1 f_2 = 0 \cdot 1 + 1 \cdot 1 = 1 = f_2^2.$$
This establishes the claim for $n = 1$. Now assume the claim is true for $n = k$, where $k \geq 1$ is some positive integer. Using this inductive hypothesis and the definition of Fibonacci numbers, we have
$$\begin{aligned} & f_0 f_1 + f_1 f_2 + ... + f_{2k-1} f_{2k} + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\ = & f_{2k}^2 + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\ = & f_{2k}(f_{2k} + f_{2k+1}) + f_{2k+1} f_{2k+2} \\ = & f_{2k} f_{2k+2} + f_{2k+1} f_{2k+2} \\ = & (f_{2k} + f_{2k+1}) f_{2k+2} \\ = & f_{2k+2} f_{2k+2} \\ = & f_{2k+2}^2. \end{aligned}$$
This establishes the claim for $n = k+1$. Having completed both the base case $n = 1$ and the inductive step, we conclude that the claim holds for all positive integers n.

6. The set of solutions is the empty set. Indeed, suppose $x = 6a + 2 = 9b + 3$ for some integers a and b. Then $3 \cdot (2a - 3b) = 6a - 9b = 3 - 2 = 1$. Hence three times an integer equals 1. This is impossible, so there are no solutions.

7. Let E be the event that $x_1 = 1$, and F be the event that $x_1 = 1$ or $x_2 = 1$. We want to find the conditional probability $P(E|F) = P(E \cap F)/P(F)$. There are $\binom{11}{3} = 165$ nonnegative integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 8$. $P(E \cap F)$ is just the probabilty that $x_1 = 1$. Since there are $\binom{9}{2} = 36$ nonnegative integer solutions to $x_2 + x_3 + x_4 = 7$, this is equal to $36/165$. To find $P(F)$, we note that by inclusion-exclusion, there are $\binom{9}{2} + \binom{9}{2} - \binom{7}{1} = 65$ solutions where $x_1 = 1$ or $x_2 = 1$. Hence $P(F) = 65/165$ and so $P(E|F) = 36/65$.

8. The characteristic polynomial for this recurrence relation is

$$r^3 - 2r^2 - r + 2 \;=\; (r+1)(r-1)(r-2)$$

The characteristic roots are $r = -1$, $r = 1$, and $r = 2$. Hence the solutions to this recurrence are of the form

$$a_n = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot 1^n + \alpha_3 \cdot 2^n.$$

To find the constants α_1, α_2, and α_3, we'll use the initial conditions. Plugging in $n = 0$, $n = 1$, and $n = 2$, we have

$$a_0 = 1 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 0 = -\alpha_1 + \alpha_2 + 2\alpha_3$$

$$a_2 = 7 = \alpha_1 + \alpha_2 + 4\alpha_3.$$

Subtracting the first equation from the third gives that $6 = 3\alpha_3$, so $\alpha_3 = 2$. The first two equations then become

$$-1 = \alpha_1 + \alpha_2$$

$$-4 = -\alpha_1 + \alpha_2.$$

Adding these two equations gives $-5 = 2\alpha_2$, so $\alpha_2 = -5/2$. Subtracting the second equation from the first gives $3 = 2\alpha_1$, so $\alpha_1 = 3/2$. Hence

$$a_n \;=\; 3/2 \cdot (-1)^n - 5/2 \cdot 1^n + 2 \cdot 2^n,$$

which we may rewrite as

$$a_n \;=\; 2^{n+1} \;+\; (-1)^n \cdot 3/2 \;-\; 5/2$$

9. The set $\mathcal{B}$ of bit strings with the same number of zeros and ones can be defined recursively as follows.

   (a) The empty string λ is in $\mathcal{B}$.

   (b) If x is in $\mathcal{B}$ then $0x1$ is in $\mathcal{B}$.

   (c) If x is in $\mathcal{B}$ then $1x0$ is in $\mathcal{B}$.

   (d) If x and y are in $\mathcal{B}$ then xy is in $\mathcal{B}$.

By structural induction, we can see that every string that the set $\mathcal{B}$ defined above has the same number of zeros and ones. We must prove the reverse inclusion. Let w be any string that has the same number of zeros and ones. We may assume that w has length at least 2, by (a). If the first and last letter in w are different then structural induction based on cases (b) or (c) shows that w lies in $\mathcal{B}$. Hence suppose that w starts and ends with the same letter. In that case we claim that $w = xy$ for some strings x and y with the same number of zeros and ones. It suffices to show this if w begins and ends with 0. We examine all proper initial substrings of w from left to right and we count the number of zeros minus the number of ones. This function starts at 1, it ends on -1, and it goes up or down by 1 in each step as we go from left right. Hence the function is 0 for some substring. This gives the desired partition $w = xy$, and the proof is complete.

10. Each child receives either two, three or four balloons. The desired number is the coefficient of x^{10} in $(x^2 + x^3 + x^4)^4$. It is found to be 10. So, there are 10 ways of distributing the balloons to the four children.