

Math 55 Midterm Exam #2

Summer 2014

Name: _____

Problem	Score	Possible
1		20
2		20
3		20
4		20
5		20
Σ		100

Note: Your answers on this test may include expressions of the form $n!$, n^k , and $\binom{n}{k}$. For example, if an answer is $52!$, you may leave it in that form; you don't need to write it out as a 68-digit number.

1. Suppose we have an unlimited supply of 4-cent and 15-cent stamps.
 - (a) Prove that we can't make 41 cents postage.

(b) Prove that we can make n cents postage for all integers $n \geq 42$.

2. 60% of all e-mail Mira receives is spam. If a message is spam, there's a 3% probability that it contains the word "opportunity". If a message is not spam, there's a 0.5% probability that it contains the word "opportunity".

What is the probability that a message containing the word "opportunity" is spam?

3. In a standard deck of 52 cards, there are 13 *ranks* (labeled 2, 3, ..., 10, J, Q, K, A) and 4 *suits* ($\clubsuit$, $\diamondsuit$, $\heartsuit$, $\spadesuit$). There is one card of each rank-suit combination (e.g., $Q\heartsuit$ is a card). A hand is a subset of the deck; the order of cards in a hand does not matter.

(a) How many 5-card hands are there in which no rank is repeated?

(b) In total, how many 1- to 13-card hands are there in which no rank is repeated?
Express your answer in as simple a form as possible.

(Continued.)

(c) What is the probability that a random 5-card hand contains at least one heart ($\heartsuit$)?

(d) What is the expected number of different suits in a random 5-card hand?

4. Prove the identity $k\binom{n}{2} + \binom{k}{2}n^2 = \binom{kn}{2}$

(a) ... algebraically.

(b) ... by a combinatorial argument.

5. For integers $n \geq 1$, let a_n be the number of subsets of $\{1, 2, 3, \dots, n\}$ that do not contain two consecutive integers. (Include the empty set.)

Determine sufficient initial values and a recurrence relation for the sequence $\{a_n\}_{n=1}^{\infty}$. Justify your answer.